Dynamics of on-line learning in two-layer neural networks in the presence of concept drift

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• Concept drift
• Model-scenario: student-teacher setup
• Including concept drift and weight decay
• Results for the ReLU- and Erf SCM, similarities and differences
Learning under concept drift

Traditional assumption in ML of stationarity is often violated

- **Virtual drift**: Change in input density $p(\xi)$
- **Real drift**: Change of the target rule $y = f(\xi)$

$$\exists \xi : p_{t_0}(\xi, y) \neq p_{t_1}(\xi, y)$$

*J. Gama, I. Žliobaitė, A. Bifet, M. Pechenizkiy, A. Bouchachia. 2014. A survey on concept drift adaptation. ACM Comput. Surv. 46, 4, Article 44 (April 2014)*
Model scenario: On-line learning of a drifting rule

**Student hypothesis**

\[ \sigma(\xi) = \sum_{i=1}^{K} g(h_i), \text{ with } h_i = w_i \cdot \xi \]

\[ K = 2 \]

**Teacher rule**

\[ \tau(\xi) = \sum_{m=1}^{M} g(b_m), \text{ with } b_m = B_m \cdot \xi \]

\[ M = 2 \]

- **Order parameters**: 
  \[ Q_{ik} = w_i \cdot w_k, \quad R_{im} = w_i \cdot B_m, \quad T_{nm} = B_n \cdot B_m = \delta_{nm} \]

- **The student learns from random i.i.d. examples** 
  \[ (\xi^\mu \in \mathbb{R}^N, \tau(\xi^\mu) \in \mathbb{R}) \]
  \[ \langle \xi_i \rangle = 0, \quad \langle \xi_i \xi_j \rangle = \delta_{ij} \]

- **CLT for large N**: 
  \[ (h_i, b_m) \text{ are zero-mean correlated Gaussian variables with } \]
  \[ \langle h_i h_k \rangle = Q_{ik}, \quad \langle b_n b_m \rangle = T_{nm}, \quad \langle h_i b_m \rangle = R_{im} \]
Learning dynamics of the networks (stationary)

Order parameters: \( Q_{ik} = w_i \cdot w_k, \quad R_{im} = w_i \cdot B_m, \quad T_{nm} = B_n \cdot B_m = \delta_{nm} \)

A stream of random i.i.d. examples \( \xi^1, \xi^2, \xi^3, \ldots \) (discrete time \( \mu = 1, 2, 3, \ldots \))

At the presentation of one example \( \xi^\mu \)

1. Quadratic error: \( \epsilon^\mu = \frac{1}{2}(\sigma^\mu - \tau^\mu)^2 \)
2. Update student weights with gradient descent:

\[
    w_i^{\mu+1} = w_i^{\mu} + \frac{n}{N} \rho_i^\mu \xi^\mu, \quad \text{with} \quad \rho_i^\mu = (\tau^\mu - \sigma^\mu)g'(x_i^\mu)
\]

3. Recursions of order parameters and example average

\[
    R_{im}^{\mu+1} = R_{im}^\mu + \frac{n}{N} \langle \rho_i^\mu b_m^\mu \rangle
\]

\[
    Q_{ik}^{\mu+1} = Q_{ik}^\mu + \frac{n}{N} \langle h_i^\mu \rho_k^\mu \rangle + \frac{n}{N} \langle h_k^\mu \rho_i^\mu \rangle + \frac{n^2}{N} \langle \rho_i^\mu \rho_k^\mu \rangle \quad \text{(Saad & Solla, 95)}
\]

4. Consider the limit \( N \to \infty \) and \( \eta \to 0 \) and scaled time:

\[
    \tilde{\alpha} = \eta \mu / N \quad d\tilde{\alpha} = \eta / N \quad \text{(continuous in the limits)}
\]

\[
    \left[ \frac{dR_{im}}{d\tilde{\alpha}} \right]_{\text{stat}} = \langle \rho_i b_m \rangle
\]

\[
    \left[ \frac{dQ_{ik}}{d\tilde{\alpha}} \right]_{\text{stat}} = \langle h_i \rho_k \rangle + \langle h_k \rho_i \rangle
\]
Introducing a random real drift

- Random change of the teacher vectors, while keeping orthonormality

\[ B_{m}^{\mu+1} \cdot B_{m}^{\mu} = 1 - \tilde{\delta}/N \]
\[ T_{nm}^{\mu+1} = B_{n}^{\mu+1} \cdot B_{m}^{\mu+1} = \delta_{nm} \]

- Weight decay as a mechanism of forgetting older examples

\[ w_{i} = (1 - \tilde{\gamma}/N) w_{i} \]

\[
\begin{align*}
\left[ \frac{dR_{im}}{d\alpha} \right]_{\text{drift}} &= \left[ \frac{dR_{im}}{d\alpha} \right]_{\text{stat}} - (\tilde{\delta} + \tilde{\gamma}) R_{im} \\
\left[ \frac{dQ_{ik}}{d\alpha} \right]_{\text{drift}} &= \left[ \frac{dQ_{ik}}{d\alpha} \right]_{\text{stat}} - 2\tilde{\gamma} Q_{ik}
\end{align*}
\]

Generalization error:

\[ \epsilon_{g} = \frac{1}{2} \langle (\sigma^{\mu} - \tau^{\mu})^2 \rangle \]

Closed form expression \( \epsilon_{g}(R_{im}, Q_{ik}) \) available for ReLU and Erf
Initial conditions corresponding to no prior information about the rule:

\[ R_{im} \approx 0 \text{ and choose } Q_{11} = Q_{22} = 0.5, \ Q_{12} = 0.49 \]

Dots: simulations for \( N = 500, \ \eta = 0.05 \) (avg. of 10 runs)
Sensitivity to drift

Erf-SCM

Plateau and final eg

Plateau length

ReLU-SCM

Plateau and final eg

Plateau length
Effects of weight decay

Erf-SCM \( \tilde{\delta} = 0.03 \)

Plateau and final eg

Plateau length

ReLU-SCM \( \tilde{\delta} = 0.2 \)

Plateau and final eg

Plateau length
In the ReLU SCM, weight decay also optimizes the specialization.
Common to both SCM...

- In the presence of concept drift, specialization possible up till $\tilde{\delta_c}$
- Drift increases the length of the plateau
- Weight decay could improve the final generalization error.

Differences between the SCM...

- Weight decay increased specialization for the ReLU SCM, while it always deteriorated specialization in the Erf SCM.
- Weight decay reduces the plateau length for the ReLU SCM, while it increases the plateau length in the Erf SCM.
- Other types of real drift, e.g. a changing complexity of the rule by (de)-aligning teacher vectors
- Virtual drift by changing the density of the input data
- Increasing number of hidden units, mismatched student and teacher.
- Universal approximators: Adaptive thresholds and hidden to output weights
- Deep networks, tree-like architectures